Trace of a linear operator

The trace of a square matrix

$A = (a_{ij}) \in \mathrm{Mat}_{n \times n}(\mathbb{C})$ is

$$\mathrm{Tr}(A) := \sum_{i=1}^{n} a_{ii} = a_{11} + \cdots + a_{nn} = \sum_{i,j=1}^{n} a_{ij}\delta_{ij}$$

Fact: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$\Rightarrow P \in GL_n(\mathbb{C})$, $\mathrm{Tr}(PAP^{-1}) = \mathrm{Tr}(A)$

Let $T \in \mathrm{Hom}(V,V)$, $\dim V < \infty$

The trace of T is defined:

Choose basis $B = \{v_1, \ldots, v_n\}$ of V

write $A = [T]_B \in \mathrm{Mat}_{n \times n}(\mathbb{C})$.

Define: $\mathrm{Tr}(T) := \mathrm{Tr}(A)$

Show: does not depend on choice of B.

If $B' = \{v_1', \dots, v_n'\}$ basis

then $[T]_{B'} = P[T]_B P^{-1}$ for some $P \in GL_n(\mathbb{C})$

$\Rightarrow$ Trace are same.

Prop: $T \in Hom(V,V)$, $S \in Hom(W,W)$

$\dim V, \dim V < \infty$, if

$U: V \to W$ linear such that $UTU^{-1} = S$

then $Tr(T) = Tr(S)$

Proof: basis $B = \{v_1, \dots, v_n\}$ of V

define $B' = \{v_1', \dots, v_n'\}$, $v_k' := Uv_k$

basis of W

$\Rightarrow$ $[S]_{B'} = [T]_B$ $\Rightarrow$ Traces are equal.

Prop: Let $T_k : V_k \to V_k$, $k=1,\dots,r$

$\dim V_k < \infty$

Let $V := V_1 \oplus \cdots \oplus V_r$, define

$T : V \to V$ by $T := T_1 \oplus \cdots \oplus T_r$

$[\, T(v_1,\dots,v_r) = (T_1(v_1),\dots,T_r(v_r)) \,]$

Then $\operatorname{Tr}(T_1 \oplus \cdots \oplus T_n) = \operatorname{Tr}(T_1) + \cdots + \operatorname{Tr}(T_r)$

Proof: Choose basis B of $V = \underline{V_1} \oplus \cdots \oplus V_r$

so that $A = [T]_B$ is a block matrix

$$A = \begin{bmatrix} A_1 & 0 & & 0 \\ 0 & A_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & A_r \end{bmatrix}, \qquad A_k = [T_k]_{B_k}$$

$\Longrightarrow \operatorname{Tr} A = \operatorname{Tr} A_1 + \cdots + \operatorname{Tr} A_r.$

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